REGULARITY OF ULTRAFILTERS

BY

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ABSTRACT

If there are κ^{++} eventually different functions from κ^+ into κ or if there are κ^{+++} eventually different functions from κ^+ into κ^+ then uniform ultrafilters on κ^+ are (κ, κ^+)-regular.

The question whether all uniform ultrafilters over κ are regular was raised by Gillman (according to [2]) and by Keisler (see [6, question 3B]). Silver showed that if κ is real-valued measurable the answer to the question is negative. The most interesting case of the question, however, is when $\kappa = \omega_1$. In this case the affirmative answer can (at the present) be a theorem of ZFC. Prikry showed [9] that in L the question is decided positively for $\kappa = \omega_1$. Chang generalized the combinatorial principle used in [9] to a principle which yields positive answer for any $\kappa < \omega_\omega$; in [4] Jensen proved that the principle is true in L. Our results are also in the positive direction and show that, for example, the weak Kurepa's hypothesis or the assumption that $2^\omega = \omega_1$ and $2^{\omega_1} > \omega_2$ imply regularity of all uniform ultrafilters over ω_1. Our notation is standard. D is an ultrafilter over the set I $(= \bigcup D)$ unless otherwise indicated. κ, λ, μ stand for infinite cardinals and $S_\kappa(\lambda) = \{X \subseteq \lambda \mid |X| < \kappa\}$. Section 1 contains results of Benda while results in Section 2 are due to Ketonen.

DEFINITION. An ultrafilter D on I is said to be (κ, λ)-regular if there is $E \subseteq D$ $|E| = \lambda$, and intersections of κ members of E are empty. D is λ-regular if it is (ω, λ)-regular.

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An equivalent and usually more useful definition (see, for example, [1]) of (κ, λ)-regularity is the following:

There is a $\tau: I \to S_\kappa(\lambda)$ which covers λ

where τ covers λ means that $\{i \mid \alpha \in \tau(i)\} \in D$ for every $\alpha < \lambda$. To show the equivalence, assume $E = \{E_\alpha \mid \alpha < \lambda\}$ is given, as in the definition. Define $\tau(i)$ for $i \in I$ as $\{\alpha < \lambda \mid i \in E_\alpha\}$. If τ is given let E_α be $\{i \mid \alpha \in \tau(i)\}$. We now prove a simple result which has been overlooked by some.

Proposition 1.1. *Let $\kappa = \mathrm{cf}(\mu) < \mu$. If for every λ, $\kappa \leqq \lambda < \mu$, an ultrafilter D is (κ, λ)-regular then it is (κ, μ)-regular.*

Proof. For every λ $(\kappa \leqq \lambda < \mu)$ let $\tau_\lambda: I \to S_\kappa(\lambda)$ be a function which covers λ. Let g be an increasing function on κ whose values are cardinals smaller than μ and whose range is cofinal in μ. Let X_α be $\{i \mid \tau_\kappa(i) \cap (\kappa - \alpha) \neq 0\}$. Then $X_\alpha \subseteq X_\beta$ if $\alpha \geqq \beta$, $X_\alpha \in D$, and $\bigcap_{\alpha < \kappa} X_\alpha = 0$, since κ is regular. Thus there is a partition of I $\{P_\alpha \mid \alpha < \kappa\}$ such that $\bigcup_{\alpha < \beta} P_\alpha \notin D$ for any $\beta < \kappa$. If $i \in P_\alpha$, define

$$\tau(i) = \bigcup_{\beta < \alpha} \tau_{g(\beta)}(i).$$

Then $\tau: I \to S_\kappa(\mu)$. τ covers μ since, if $\gamma < \mu$, then $\gamma < g(\beta)$ for some $\beta < \kappa$ and we then have

$$\{i \mid \gamma \in \tau(i)\} \supseteq \{i \mid \gamma \in \tau_{g(\beta)}(i)\} \cap (I - \bigcup_{\gamma < \beta} P_\gamma).$$

Remark. Proposition 1.1 together with [10, Th. 20] imply that in L every uniform ultrafilter over regular and not weakly compact cardinal κ is λ-regular where $\lambda = \min(\kappa, \omega_\omega)$.

We now proceed to give still another definition of (κ, λ)-regularity which, though more awkward than the usual one, will be better suited for our purposes. Throughout the paper, if $f, g: I \to \mathrm{ord}$ and D is an ultrafilter on I, $f/D < g/D$ means that $\{i \mid f(i) < g(i)\} \in D$.

Theorem 1.2. *(κ, λ)-regularity of ultrafilter D on I is equivalent to the following statement.*

(1) *There is a function $s: I \to S_{\kappa^+}(\lambda)$ which covers λ and there are functions $g_\alpha: I \to \kappa$ for $\alpha \leqq \lambda$ such that $g_\alpha/D < g_\lambda/D$ if $\alpha < \lambda$ and if $\alpha, \beta \in s(i)$ and $\alpha \neq \beta$ then $g_\alpha(i) \neq g_\beta(i)$.*

Proof. Assume that $\tau: I \to S_\kappa(\lambda)$ covers λ. For $\alpha < \lambda$ and $i \in I$ define

$$g_\alpha(i) = \begin{cases} \xi \text{ if } \alpha \in \tau(i) \text{ and is the } \xi\text{th element of } \tau(i) \\ \\ 0 \text{ (or undefined) if } \alpha \notin \tau(i). \end{cases}$$

Then $g_\alpha : I \to \kappa$. Since $|\tau(i)| < \kappa$, $\bigcup \{g_\alpha(i) \mid \alpha < \lambda\}$ is an ordinal less than κ, thus we can define g_λ so that $g_\lambda(i) > g_\alpha(i)$ for every $\alpha < \lambda$ and $i \in I$. Then (1) with s being τ is clearly satisfied. If, on the other hand, s and $g_\alpha(\alpha \leqq \lambda)$ are given as in (1), define for $i \in I$

$$\tau(i) = s(i) \cap \{\alpha < \lambda \mid g_\alpha(i) < g_\lambda(i)\}.$$

Then τ covers λ. If $|\tau(i)|$ were greater than or equal to κ we would have $a \subseteq s(i)$, $|a| = \kappa$ such that if $\alpha \in a$ then $g_\alpha(i) < g_\lambda(i)$. But the set $\{g_\alpha(i) \mid \alpha \in a\}$ has cardinality κ by (1) which contradicts $g_\lambda(i) < \kappa$.

An application of (1) gives the following result on stretching regularity.

THEOREM 1.3. *If D is a (κ^+, λ)-regular ultrafilter over I and* $\mathrm{cof}(\langle\kappa, <\rangle I/D) > \lambda$ *then D is (κ, λ)-regular. If κ is regular the converse is also true.*

PROOF. For each $i \in I$ let $\{\alpha^i_\xi \mid \xi < \kappa\}$ enumerate without repetitions $\tau(i)$. We can assume that $|\tau(i)| = \kappa$ for every $i \in I$ since if $\{i \in I \mid |\tau(i)| < \kappa\}$ is in D, D already is (κ, λ)-regular. Define for $\alpha < \lambda$

$$g_\alpha(i) = \begin{cases} \xi & \text{if } \alpha \in \tau(i) \text{ and } \alpha = \alpha^i_\xi \\ 0 & \text{if } \alpha \notin \tau(i). \end{cases}$$

Since $\mathrm{cof}(\langle\kappa, <\rangle I/D) > \lambda$ there is a $g_\lambda : I \to \kappa$ such that $g_{\lambda/D}$ is larger than every $g_{\alpha/D}(\alpha < \lambda)$. Then (1) is true with s equal to τ. The converse is a standard fact: if $f_\alpha : I \to \kappa(\alpha < \lambda)$ are given, define $g(i) = \sup\{f_\alpha(i) \mid \alpha \in \tau(i)\} + 1$ where $\tau : I \to S_\kappa(\lambda)$ covers λ. Then $g : I \to \kappa$ (since κ is regular) and $f_\alpha/D < g/D$ for every $\alpha < \lambda$.

A special case of the theorem, namely when $\kappa^+ = \lambda$ and D is uniform over λ and thus (λ, λ)-regular, was proved by Jorgensen using matrices. K. Kunen simplified his proof and Kunen's proof lead us to condition (1).

COROLLARY 1.4. *Let D be a uniform ultrafilter over ω_m, $m < \omega$. D is regular (that is, ω_m-regular) iff*

$$\mathrm{cof}((\omega_n, <)^{\omega_m}/D) > \omega_m \text{ for every } n < m.$$

PROOF. The implication from left to right was proved above. For the other implication, note that D is (ω_m, ω_m)-regular. Theorem 2 allows us to stretch, under the assumptions, (ω_n, ω_m)-regularity to (ω_{n-1}, ω_m)-regularity.

Condition (1) has the disadvantage of depending on D. We will now state a stronger, uniform form of (1) which in a special case is a consequence of Kurepa's hypothesis. K. Kunen has coined, for the condition, the phrase *weak Kurepa's hypothesis*, hence we make the following definition.

DEFINITION. Let F be a filter on κ^+. We say that the weak Kurepa's hypothesis holds for F (for short we shall write $\mathrm{wKH}_{\kappa^+}(F)$) if there is a set $\{f_\alpha \mid \alpha < \kappa^{++}\}$ such that $f_\alpha : \kappa^+ \to \kappa$ and $\{\xi < \kappa^+ \mid f_\alpha(\xi) \neq f_\beta(\xi)\} \in F$ whenever $\alpha < \beta < \kappa^{++}$. If F_0 is the filter $\{X \subseteq \kappa^+ \mid |\kappa^+ - X| \leqq \kappa\}$, we write wKH_{κ^+} for $\mathrm{wKH}_{\kappa^+}(F_0)$.

REMARKS. If $F \subseteq F'$ then $\mathrm{wKH}(F)$ implies $\mathrm{wKH}(F')$. If D is an ultrafilter on κ^+ then $\mathrm{wKH}_{\kappa^+}(D)$ iff $|\kappa^{\kappa^+}/D| \geqq \kappa^{++}$.

The generalized Kurepa's hypothesis was defined by Jensen in [4]. We restate it as follows.

DEFINITION. $\mathrm{KH}_{\kappa,\lambda}$ holds if there is a one-to-one function $A : \lambda^+ \to P(\lambda)$ such that for every infinite $x \leqq \lambda$, $|x| < \kappa$, $|\{A_\alpha \cap x \mid \alpha < \lambda^+\}| \leqq |x|$. By KH_κ we mean that there is a one-one function $A : \kappa^+ \to P(\kappa)$ such that for every infinite $\alpha < \kappa$, $|\{A_\beta \cap \alpha \mid \beta < \kappa\}| \leqq |\alpha|$.

REMARKS. Solovay (in an unpublished paper) proved that if $\kappa \geqq \omega$, $A \subseteq \kappa^+$. and $V = L[A]$ hold then KH_{κ^+} holds. Jensen [2] proved that in L, $\mathrm{KH}_{\kappa,\kappa}$ holds iff κ is not ineffable. The following proposition is a well-known fact.

PROPOSITION 1.5. KH_{κ^+} *implies* wKH_{κ^+}.

PROOF. Let $A : \kappa^{++} \to P(\kappa^+)$ be as in the definition of KH_{κ^+}. For every $\beta < \kappa^+$ let $\{A^\beta_\xi \mid \xi < \kappa\}$ enumerate $\{A_\alpha \cap \beta \mid \alpha < \kappa^{++}\}$. Define for $\beta < \kappa^+$, $f_\alpha(\beta) = \xi$ iff $A_\alpha \cap \beta = A^\beta_\xi$. Then if $\alpha \neq \alpha'$ and β is such that $\beta \in A_\alpha$ iff $\beta \notin A_{\alpha'}$, then $f_\alpha(\gamma)$ and $f_{\alpha'}(\gamma)$ differ for every $\gamma \geqq \beta$.

An apparently weaker statement than wKH_{κ^+} is $\mathrm{wKH}_{\kappa^+}(C)$ where C is the filter generated by closed unbounded subsets of κ^+. But, as J. Silver has remarked, $\mathrm{wKH}_{\omega_1}(C)$ is still strong enough to refute Chang's conjecture for (ω_2, ω_1), (ω_1, ω_0). In brief, his argument runs as follows. Form a structure whose universe contains the functions for $\mathrm{wKH}_{\omega_1}(C)$ and all countable ordinals. Add unary predicates U_0, U_1 for ω_0, ω_1 respectively. Add also a ternary predicate $R(f, g, \alpha)$ so that if $f \neq g$ then $X = \{\alpha < \omega_1 \mid R(f, g, \alpha)\} \in C$ and f, g differ on X. Finally add a few more predicates to ensure the interpretation of U_1 in every elementary substructure as a limit ordinal. If this model had an elementary substructure of type (ω_1, ω), we would have ω_1 functions from a countable ordinal α (equal to the interpretation of U_1) into ω. Using properties of $R(\cdot,\cdot,\cdot)$ it is not difficult to see that any two functions in the elementary substructure would differ at α.

(1) and wKH_{κ^+} give the following criterion for (κ,κ^+)-regularity.

THEOREM 1.6. *If F is a κ^+-complete filter on κ^+ and $\mathrm{wKH}_{\kappa^+}(F)$ holds then every ultrafilter on κ^+ which extends F is (κ,κ^+)-regular.*

PROOF. Let $\{f_\alpha \mid \alpha < \kappa^{++}\}$ satisfy $\mathrm{wKH}_{\kappa^+}(F)$. Let $D \supseteq F$ be an ultrafilter. The set $\{f_\alpha/D \mid \alpha < \kappa^{++}\}$ has power κ^{++} and is linearly ordered by $<$. There must exist $\alpha < \kappa^{++}$ such that $\{\xi \mid f_\xi/D < f_\alpha/D\}$ has power κ^+. We can assume that if $\alpha < \kappa^+$ then $f_\alpha/D < f_{\kappa^+}/D$. Let $\tau(i)$ be

$$\{\alpha < \kappa^+ \mid \forall \xi < \alpha)[f_\xi(i) \neq f_\alpha(i)] \wedge f_\alpha(i) < f_{\kappa^+}(i)\}.$$

We show that $\tau: I \to S_\kappa(\kappa^+)$. Indeed, if $|\tau(i)| \geqq \kappa$ then $|\{f_\alpha(i) \mid \alpha \in \tau(i)\}| \geqq \kappa$ which contradicts $\{f_\alpha(i) \mid \alpha \in \tau(i)\} \subseteq f_{\kappa^+}(i) < \kappa$. τ covers κ^+ since if $\alpha < \kappa^+$ is given, then

$$\{i \mid \alpha \in \tau(i)\} \supseteq \{i \mid f_\alpha(i) < f_{\kappa^+}(i)\} \cap \bigcap_{\xi < \alpha} \{i \mid f_\xi(i) \neq f_\alpha(i)\}.$$

The right-hand side of the inclusion is in D because $\bigcap_{\xi<\alpha} \cdots \in F$ and $F \subseteq D$.

COROLLARY 1.7. *wKH_{ω_1} implies that every uniform ultrafilter on ω_1 is regular. If there is a nonregular ultrafilter over ω_1 then ω_2 is strongly inaccessible in L.*

PROOF. An ultrafilter D on ω_1 is uniform iff it extends the filter $\{X \subseteq \omega_1 \mid |\omega_1 - X| < \omega_1\}$ and this filter is ω_1-complete. The second statement follows from the well-known fact that KH_{ω_1} holds if ω_2 is not inaccessible in L.

This result should be compared with [9, Th. 3]. There Prikry has proved Corollary 1.7 assuming a combinatorial principle stronger than KH_{ω_1}. (Whether it is actually stronger is open.) Nevertheless Baumgartner proved that if Con(ZFC + some inaccessible cardinal) then Con(ZFC + wKH_{ω_1} + $\neg\mathrm{KH}_{\omega_1}$).

REMARK. The preceeding discussions show that irregular ultrafilters are more difficult to obtain than was previously thought. In the proof of Theorem 1.6 we did not need full wKH. We used it only for the purpose of obtaining for some given D on ω_1, ω_1 eventually different functions (that is, any two of which differ from each other after some point) from ω_1 into ω bounded mod D by a function from ω_1 into ω. Now there are many eventually different families of such functions. For example, define for $\alpha, \xi < \omega_1$, $f_\alpha(\xi) = \alpha$, if $\xi > \alpha$ and $f_\alpha(\xi) = 0$ if

$\xi \leqq \alpha$. Since $f_\alpha(\xi) < \xi < \omega_1$, f_α can be assumed to be from ω_1 into ω. Clearly, if $\alpha \neq \beta$ then f_α, f_β differ after $\max(\alpha, \beta)$. The proof of Theorem 1.6 implies that any such family must be cofinal in $(\omega, <)^{\omega_1}/D$. Consequently, every such family must be ω_1-like ordered mod D, that is, below any element of such a family there can be at most countably many members of the family. In general we can state the following corollary.

COROLLARY 1.8. *If κ is regular and D, an ultrafilter on κ^+, is not (κ,κ^+)-regular then* $\mathrm{cof}((\kappa, <)^{\kappa^+}/D) = \kappa^+$.

By Theorem 1.6 Kurepa's hypothesis can be applied to stretch (κ,κ)-regularity (that is, uniformity) to gap one-regularity. It is natural to ask whether the generalized Kurepa's hypothesis we mentioned implies more. As we have shown, it does for certain ultrafilters, in fact for those which should be the most troublesome. Following [3] we define the notion of closed unbounded subsets of $S_\kappa(\lambda)$ (see also [8]).

DEFINITION. A subset $X \subseteq S_\kappa(\lambda)$ is unbounded if for any $s \in S_\kappa(\lambda)$ there is an $r \in X$ such that $s \subseteq r$. X is closed if, for any $\{s_\alpha \mid \alpha < \mu\} \subseteq X$ where $\mu < \kappa$ and $s_\alpha \subseteq s_\beta$, we have $\bigcup_{\alpha<\mu} s_\alpha \in X$ if $\alpha < \beta < \mu$.

It is known (see, for example, [3]) that if κ is regular the filter $F_{\kappa,\lambda}$ generated by closed unbounded subsets of $S_\kappa(\lambda)$ is κ-complete and normal. That is, if $\{X_\alpha \mid \alpha < \lambda\} \subseteq F_{\kappa,\lambda}$ then

$$\{s \in S_\kappa(\lambda) \mid (\forall \alpha \in s)[s \in X_\alpha]\} \in F_{\kappa,\lambda}.$$

We need a stronger form of normality than the one usually stated.

LEMMA 1.9. *Let κ be regular, $\kappa \leqq \lambda$ and $n < \omega$. Let $\{X_r \mid r \subseteq \lambda \wedge |r| = n\} \subseteq F_{\kappa\lambda}$. Then*

$$X = \{s \in S_\kappa(\lambda) \mid (\forall r \subseteq s)[|r| = n \rightarrow s \in X_r]\} \in F_{\kappa\lambda}.$$

PROOF. If $\{s_\xi \mid \xi < \mu\} \subseteq X$ is strictly $\subseteq$-increasing, $\omega \leqq \mu < \kappa$ and $r \subseteq s = \bigcup_{\xi<\mu} s_\xi$ where $|r| = n$, then $r \in s_\xi$ for some $\xi < \mu$. If $\zeta \geqq \xi$ then $s_\zeta \in X_r$, since $s_\zeta \in X$, and thus $s = \bigcup_{\xi \leqq \zeta} s_\zeta \in X_r$ since X_r is closed. Thus X is closed.

To show that X is unbounded, let $r \in S_\kappa(\lambda)$. Define by induction $r_0 = r$ and r_{k+1} such that $r_{k+1} \supseteq r_k$ and $r_{k+1} \in \bigcap \{X_r \mid |r| = n \wedge r \subseteq r_k\}$. r_{k+1} exists since $|\{r \mid |r| = n \wedge r \subseteq r_k\}| < \kappa$. Then $\bigcup_{k<\omega} r_k \in X$.

The weak version of $KH_{\kappa,\lambda}$ may be stated as follows.

DEFINITION. $wKH_{\kappa,\lambda}$ holds if there are functions $f_\alpha(\alpha < \lambda^+)$ defined on $S_\kappa(\lambda)$ such that for every infinite $s \in S_\kappa(\lambda)$, $f(s) < |s|$ and if $\alpha \neq \beta$ then for some $\gamma < \lambda$, f_α and f_β differ on $\{s \in S_\kappa(\lambda) \mid \gamma \in s\}$. One can prove, as above, that $KH_{\kappa,\lambda}$ implies $wKH_{\kappa,\lambda}$.

THEOREM 1.10. *Let $wKH_{\kappa^+,\lambda}$ hold and let D be an ultrafilter on $S_\kappa(\lambda)$ which extends the filter $F_{\kappa,\lambda}$. Then D is (κ,λ)-regular.*

PROOF. The assumption implies that D is (κ^+,λ)-regular (take for τ the identity). We again have λ^+ function into ordinals so we may assume that $f_\alpha/D < f_\lambda/D$ for every $\alpha < \lambda$. For $\alpha < \beta < \lambda$ let $h(\alpha,\beta)$ be the least γ such that $f_\alpha(s) \neq f_\beta(s)$ for every s containing γ. We are looking for sets which are closed under h. Hence let $X_{\{\alpha,\beta\}}$ be $\{s \in S_\kappa(\lambda) \mid \alpha, \beta, h(\alpha,\beta) \in s\}$. $X_{\{\alpha,\beta\}} \in F_{\kappa,\lambda}$, so by Lemma 1.9

$$X = \{s \mid (\alpha, \beta \in s)[\alpha \neq \beta \rightarrow s \in X_{\{\alpha,\beta\}}]\}$$

is in $F_{\kappa,\lambda}$; hence $X \in D$. For $s \in S_\kappa(\lambda)$ define

$$\tau(s) = \begin{cases} \{\alpha \in s \mid f_\alpha(s) < f_\lambda(s)\} & \text{if } s \in X \\ 0 & \text{if } s \notin X. \end{cases}$$

It is obvious that τ covers λ. If $|\tau(s)| \geqq \kappa$ then $s \in X$, thus if $\alpha, \beta \in \tau(s) \subseteq s$, $\alpha \neq \beta$, then $h(\alpha,\beta) \in s$ which means that $f_\alpha(s) \neq f_\beta(s)$. Then the set $\{f_\alpha(s) \mid \alpha \in \tau(s)\}$ has power κ and is included in $f_\lambda(s) < |s| \leqq \kappa$ which is a contradiction.

REMARK. The result can be generalized in different ways. For example, assume that D is (κ^+,λ)-regular over λ via a covering function $\tau: \lambda \rightarrow S_{\kappa^+}(\lambda)$ for which $\tau^{-1}(X) \in D$ for every closed unbounded $X \subseteq S_\kappa(\lambda)$. However, this assumption implies that $\lambda^\kappa = \lambda$. If we go to a special case $\lambda = \omega_2$ and assume KH_{ω_2,ω_2}, then by Theorem 1.6 every uniform ultrafilter D on ω_2 is (ω_1,ω_2)-regular. Thus some $\tau: \omega_2 \rightarrow S_{\omega_1}(\omega_2)$ covers ω_2. But it might happen that $\tau(\alpha)$ would have order type ω for every $\alpha < \omega_2$. We feel that in this case D would be closer to regularity than in the case when $\tau^{-1}(X) \in D$ for every $X \in F_{\omega_1,\omega_2}$.

In general one would think that irregular ultrafilters contain large complete filters. Specifically, if D is uniform and not regular over ω_1, does it contain a set together with the filter of closed unbounded subsets of the set?

The above suggests a finer definition of regularity, namely that D is (α,κ)-regular (where α is an ordinal) if there is a $\tau: I \rightarrow S_\alpha(\kappa)$ where $S_\alpha(\kappa)$ are subsets of κ of order type less that α. One may then ask: is every uniform ultrafilter over ω_1 $(\omega+1,\omega_1)$-regular?

To explain the hypothesis of Theorem 1.10 note that, by analogy with the proof of Theorem 1.6, we need the following: $\{i \mid \tau(i) \text{ is closed under } h\} \in D$. Since λ itself is closed under h and since the theory of supercompact cardinals shows that $F_{\kappa\lambda}$ closely approximates λ (see also [8]), $F_{\kappa,\lambda}$ comes in naturally.

Finally, let us mention a type of ultrafilters which appeared in an earlier attempt to improve Theorem 1.6. We call an ultrafilter D on I $[\kappa]^2$-regular if there is $\{X_s \mid s\in[\kappa]^2\} \subseteq D$ such that $\bigcap \{X_s \mid s \in [S]^2\} = 0$ for every infinite $S \subseteq \kappa$. It is clearly a weaker notion that κ-regularity. Moreover, by the Erdös-Rado theorem, D cannot be $[2^{|I|^+}]^2$-regular. On the other hand K. Kunen noticed that every ω-incomplete ultrafilter is $[2^\omega]^2$-regular; in fact if D is κ-regular then it is $[2^\kappa]^2$-regular. Let $\tau: I \to S_\omega(\kappa)$ cover κ. Define, for $U, V \subseteq \kappa, U \neq V$

$$X_{\{U,V\}} = \{i \mid \tau(i) \cap U \neq \tau(i) \cap V\}.$$

We do not know how regular a $[2^\kappa]^2$-regular ultrafilter is.

2.

We have seen that if there are κ^{++} eventually different functions from κ^+ into κ then ultrafilters on κ^+ are (κ,κ^+)-regular. The next theorem arrives at the same conclusions starting from a different hypothesis.

THEOREM 2.1. *Let there be κ^{+++} eventually different functions from κ^+ into κ^+. Then every uniform ultrafilter over κ^+ is (κ,κ^+)-regular.*

PROOF. Let $\{f_\alpha \mid \alpha < \kappa^{+++}\} \subseteq {}^{\kappa^+}\kappa^+$ be such a family and let D be a uniform ultrafilter over κ^+. We can assume that for every $\alpha < \kappa^{++}$, $f_\alpha/D < f_{\kappa^{++}}/D = g/D$. Let $\{\gamma_\alpha \mid \alpha < \kappa\}$ enumerate $g(i)(<\kappa^+)$. For $\alpha < \kappa^{++}$ define

$$g_\alpha(i) = \begin{cases} \text{least } \beta[f_\alpha(i) = \gamma^i_\beta], & \text{if } f_\alpha(i) < g(i) \\ \text{undefined if not.} \end{cases}$$

Each g_α is defined on a set from D and is into κ. $\{g_\alpha/D \mid \alpha < \kappa^{++}\}$ has power κ^{++}, so we may assume that $g_\alpha/D < g_{\kappa^+}/D$ for every $\alpha < \kappa^+$. Let $\tau(i)$ be $\{\alpha < \kappa^+ \mid g_\alpha$ defined on $i \wedge (\forall\xi < \alpha)[g_\xi(i) \neq g_\alpha(i)]\}$. Then τ covers κ^+ because D is uniform and g_α, g_β are eventually different on the set where both are defined. If $\alpha,\beta \in \tau(i)$, $\alpha \neq \beta$ then $g_\alpha(i) \neq g_\beta(i)$. Thus (R) is satisfied and hence D is (κ,κ^+)-regular.

As a consequence we obtain a cardinality assumption which guarantees regularity of ultrafilters on ω_1.

COROLLARY 2.2. *If $2^{\omega} = \omega_1$ and $2^{\omega_1} > \omega_2$ then every uniform ultrafilter over ω_1 is regular.*

PROOF. By a result in [11], we obtain from the above assumptions a family of ω_3 eventually different functions from ω_1 into ω_1.

REMARK. The same hypothesis implies that every ω_1-complete filter over ω_1 is ω_2-saturated (see [7]).

The concept of a p-point is also useful in describing a large class of regular ultrafilters.

DEFINITION. An ultrafilter D on κ^+ is called a p-point if for any partition $\{P_\alpha \mid \alpha < \kappa^+\}$ of κ^+ such that $|\{\alpha < \kappa^+ \mid X \cap P_\alpha \neq 0\}| = \kappa^+$ for any $X \in D$ one can find a $Y \in D$ such that $|Y \cap P_\alpha| < \kappa$ for any $\alpha < \kappa^+$.

THEOREM 2.3. *Every uniform ultrafilter over κ^+ which is not a p-point is (κ, κ^+)-regular.*

PROOF. Let us assume that D is uniform and not (κ, κ^+)- regular. Let $\{P_\alpha \mid \alpha < \kappa^+\}$ be a partition of κ^+ such that $|\{\alpha < \kappa^+ \mid X \cap P_\alpha \neq 0\}| = \kappa^+$ for any $X \in D$. The partition determines the function $f\colon \kappa^+ \to \kappa^+$ which is α on P_α.

We shall first show that there is an $X \in D$ so that $|X \cap P_\alpha| \leqq \kappa$ for every $\alpha < \kappa$. If this were not the case then for every $g\colon \kappa^+ \to \kappa^+$, $g \circ f < \mathrm{id}/D$ ($g \circ f$ is constant on P_α). Now, if $\{f_\alpha \mid \alpha < \kappa^{++}\}$ is a family of eventually different functions from κ^+ into κ^+ then $\{f_\alpha \circ f \mid \alpha < \kappa^{++}\}$ is a family of functions eventually different with respect to $\{P_\alpha \mid \alpha < \kappa^+\}$. That is, if $\alpha \neq \beta$ there is a γ such that $f_\alpha \circ f$ and $f_\beta \circ f$ differ on $\bigcup\{P_\xi \mid \xi \geqq \gamma\}$) which is bounded by identity.

The discussion preceeding Corollary 1.8 indicates that D would be (κ, κ^+)-regular. Thus we can assume that $|P_\alpha| \leqq \kappa$ for every $\alpha < \kappa$. Let $\{g_\alpha \mid \alpha < \kappa^+\}$ be a family of eventually different functions from κ^+ into κ and let $h\colon \kappa^+ \to \kappa$ be one-to-one on each P_α. The same argument shows that for some $\alpha_0 < \kappa^+$, $g_{\alpha_0} \circ f \geqq h$ on an $X \in D$. But it is clear from the choice of g_α's and h that $|X \cap P_\alpha| < \kappa$ for every $\alpha < \kappa$.

COROLLARY 2.4. *Every uniform non-regular ultrafilter on κ^+ is a p-point. If D is a non-regular ultrafilter extending the closed unbounded filter on κ^+ then D is weakly normal (that is, every function less than* $\mathrm{id}\,(\mathrm{mod}\, D)$ *is less than* $\alpha(\mathrm{mod}\, D)$ *for some $\alpha < \kappa^+$*).

REMARK. If $\{P_\alpha \mid \alpha < \omega_1\}$ is a partition of ω_1 such that $|P_\alpha| = \omega_1$ for every $\alpha < \omega_1$ define

$$F = \{A \subseteq \omega_1 \,|\, (\forall \alpha < \omega_1) \,|(\omega_1 - A) \cap P_\alpha| \leqq \omega\}.$$

F is an ω_1-complete filter on ω_1. It is easy to see that no ultrafilter which extends F is a p-point, therefore, by Corollary 2.2, every such ultrafilter is regular.

The next result extends Theorem 1.3 to singular cardinals.

COROLLARY 2.5. *Let λ be* cof(κ) *and let D be a uniform ultrafilter over κ^+. If D is not (κ, κ^+)-regular then* $\mathrm{cof}((\lambda, <)^{\kappa^+}/D) = \kappa^+$.

PROOF. It is straightforward to prove that $\mathrm{cof}((\lambda, <)^{\kappa^+}/D) = \mathrm{cof}((\kappa, <)^{\kappa^+}/D)$ By Corollary 1.8, $\mathrm{cof}((\kappa, <)^{\kappa^+}/D) = \kappa^+$.

Corollary 2.5 implies the following proposition.

PROPOSITION 2.6. (GCH). *Every uniform ω_1-descendingly complete ultrafilter over $\omega_{\omega+1}$ is $(\omega_\omega, \omega_{\omega+1})$-regular.*

PROOF. A result of Jorgensen [5] and Prikry [10] shows that if D is ω_1-descendingly complete overl I then $|\omega^I/D| \leqq \omega_1$. Hence $\mathrm{cof}((\omega, <)^I/D) < \omega_{\omega+1}$. Applying Corollary 2.5 we see that D must be $(\omega_\omega, \omega_{\omega+1})$-regular.

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